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OPTIMALITY AND DUALITY FOR NONDIFFERENTIABLE FRACTIONAL PROGRAMMING WITH GENERALIZED INVEXITY

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Abstract. We establish necessary and sufficient optimality conditions for a class of generalized nondifferentiable fractional optimization programming problems. Moreover, we prove the weak and strong duality theorems under (V, ρ)-invexity assumption.

1. Introduction and preliminaries

Many authors have introduced various concepts of generalized convexity and have obtained optimality and duality results for optimization programming problem ([1]-[4], [6]-[12]). Many practical problems encountered in economics, engineering design, and management science, and so forth can be described by nonsmooth functions. The theory of nonsmooth optimization using locally Lipschitz functions was introduced by Clarke [5].

We consider the following generalized nondifferentiable fractional optimization problem (GFP):

$$\begin{array}{lll}\text{(GFP)} & \text{Minimize} & \max\left\{\dfrac{f_i(x)}{g_i(x)} \mid i = 1, \cdots, p\right\} \\ & \text{subject to} & h_j(x) \leq 0, \quad j = 1, \cdots, m,\end{array}$$

where $f := (f_1, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1, \cdots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \cdots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are locally Lipschitz function. We assume that $f_i(x) \geqq 0$ and $g_i(x) > 0$, $i = 1, \cdots, p$. Let $X_0 := \{x \in \mathbb{R}^n \mid h_j(x) \leqq$

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$0,\ j = 1, \cdots, m\}$ be the feasible set of (GFP). Let $J = \{1, 2, \cdots, m\}$ and $J(x_0) = \{j \in J \mid h_j(x_0) = 0\}$.

We consider the following fractional optimization problem (FP):

$$\text{(FP)} \qquad \text{Minimize} \qquad \max\left\{\frac{f_i(x) + s(x|C_i)}{g_i(x)} \mid i = 1, \cdots, p\right\}$$
$$\text{subject to} \qquad h_j(x) \leq 0, \quad j = 1, \cdots, m,$$

where $f := (f_1, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p,\ g := (g_1, \cdots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \cdots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable function. For each $i = 1, \cdots, p$, C_i is compact convex set of $\mathbb{R}^n$ and $s(x|C_i) := \max\{\langle x, y_i\rangle \mid y_i \in C_i\}$.

Recently, Kim and Kim [7] consider the nondifferentiable fractional optimization problem (FP), in which each component of the objective function contains a term involving the support function of a compact convex set. They established necessary and sufficient optimality conditions for fractional optimization problem (FP). And they formulated a Mond-Weir type dual problem for (FP) and showed that the weak and strong duality.

In this paper, we apply the approach of Kim and Kim[7] to the generalized nondifferentiable fractional optimization problem (GFP), we establish necessary and sufficient optimality conditions for a nondifferentiable fractional optimization programming involving locally Lipschitz functions. Moreover, we prove the weak and strong duality theorems under (V, ρ)-invexity assumption.

Now we give some notations for our results in this section;

Let a function $f : \mathbb{R}^n \to \mathbb{R}$ be given. We shall suppose that f is locally Lipschitz, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U and a constant $L > 0$ such that for all y and z in U,

$$|f(y) - f(z)| \leq L\|y - z\|.$$

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of g at $a \in \mathrm{dom} g$ is defined by

$$\partial g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle \quad \forall x \in \mathrm{dom} g\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$ and $\mathrm{dom} g := \{x \in \mathbb{R}^n \ :\ g(x) < +\infty\}$.

Definition 1.1. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be (V, ρ)-invex at $u \in \mathbb{R}^n$ with respect to the function η and $\theta_i : \mathbb{R}^n \to \mathbb{R}^n$ if there exists $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1, \ldots, p$ such that for any

$\xi_i \in \partial f_i(u)$, $i = 1, \ldots, p$ and any $x \in \mathbb{R}^n$, and for all $i = 1, \ldots, p$,

$$\alpha_i(x,u)[f_i(x) - f_i(u)] \geq \xi_i^T \eta(x,u) + \rho_i \|\theta_i(x,u)\|^2.$$

LEMMA 1.2. [5] *Let f and g be Lipschitz near x and suppose that $g(x) \neq 0$. Then $\frac{f}{g}$ is Lipschitz near x, and one has*

$$\partial \left(\frac{f}{g} \right)(x) \subset \frac{g(x)\partial f(x) - f(x)\partial g(x)}{\{g(x)\}^2}.$$

If in addition $f(x) \geqq 0$, $g(x) > 0$ and if f and $-g$ are regular at x, then equality holds and $\frac{f}{g}$ is regular at x.

THEOREM 1.3. *Assume that f and g are vector-valued differentiable functions defined on $\mathbb{R}^n$ and $f(x) \geq 0$, $g(x) > 0$ for all $x \in \mathbb{R}^n$. If f and $-g$ are regular and (V, ρ)-invex at x_0, then $\frac{f}{g}$ is (V, ρ)-invex at x_0, where*

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)} \alpha_i(x, x_0), \quad \bar{\theta}_i(x, x_0) = \left(\frac{1}{g_i(x_0)} \right)^{\frac{1}{2}} \theta_i(x, x_0).$$

Proof. Let x, $x_0 \in X_0$. Then, by the (V, ρ)-invexity of f and $-g$, there exists $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1, \ldots, p$ such that for any $\xi_i \in \partial f_i(x_0)$, $\zeta_i \in \partial g_i(x_0)$, $i = 1, \ldots, p$ and $x \in \mathbb{R}^n$, and for all $i = 1, \ldots, p$,

$$\alpha_i(x, x_0)[f_i(x) - f_i(x_0)] \geq \xi_i^T \eta(x, x_0) + \rho_i \|\theta_i(x, x_0)\|^2,$$
$$\alpha_i(x, x_0)[g_i(x) - g_i(x_0)] \geq \zeta_i^T \eta(x, x_0) + \rho_i \|\theta_i(x, x_0)\|^2.$$

So, we have for any $\xi_i \in \partial f_i(x_0)$, $\zeta_i \in \partial g_i(x_0)$, $i = 1, \ldots, p$ and $x \in \mathbb{R}^n$, and for all $i = 1, \ldots, p$,

$$\begin{aligned}
&\alpha_i(x, x_0) \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \right) \\
&= \alpha_i(x, x_0) \left(\frac{f_i(x) - f_i(x_0)}{g_i(x)} - f_i(x_0) \frac{g_i(x) - g_i(x_0)}{g_i(x) g_i(x_0)} \right) \\
&\geqq \frac{\xi_i^T \eta(x, x_0) + \rho_i \|\theta_i(x, x_0)\|^2}{g_i(x)} - \frac{f_i(x_0)}{g_i(x) g_i(x_0)} \left(\zeta_i^T \eta(x, x_0) + \rho_i \|\theta_i(x, x_0)\|^2 \right).
\end{aligned}$$

Since $g_i(x) > 0$, $i = 1, \ldots, p$ for all $x \in X_0$, we have for any $\xi_i \in \partial f_i(x_0)$, $\zeta_i \in \partial g_i(x_0)$, $i = 1, \ldots, p$ and $x \in \mathbb{R}^n$, and for all $i = 1, \ldots, p$,

$$\begin{aligned}
&\alpha_i(x,x_0)\left(\frac{f_i(x)}{g_i(x)}-\frac{f_i(x_0)}{g_i(x_0)}\right)\\
&\geqq \frac{g_i(x_0)}{g_i(x)g_i(x_0)}\Big[\xi_i^T\eta(x,x_0)+\rho_i\|\theta_i(x,x_0)\|^2\Big]\\
&\quad -\frac{g_i(x_0)}{g_i(x)}\left(\frac{f_i(x_0)\zeta_i^T\eta(x,x_0)}{(g_i(x_0))^2}+\rho_i\|\left(\frac{f_i(x_0)}{(g_i(x_0))^2}\right)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\right).
\end{aligned}$$

Thus, from Lemma 1.2, for any $\omega_i \in \partial\left(\frac{f_i}{g_i}\right)(x_0)$, $\xi_i \in \partial f_i(x_0)$, $\zeta_i \in \partial g_i(x_0)$, $i=1,\ldots,p$ and $x\in\mathbb{R}^n$, and for all $i=1,\ldots,p$,

$$\begin{aligned}
&\alpha_i(x,x_0)\left(\frac{f_i(x)}{g_i(x)}-\frac{f_i(x_0)}{g_i(x_0)}\right)\\
&\geqq \frac{g_i(x_0)}{g_i(x)}\Big[\left(\frac{\xi_i g_i(x_0)-\zeta_i f_i(x_0)}{(g_i(x_0))^2}\right)^T\eta(x,x_0)+\rho_i\|\left(\frac{1}{g_i(x_0)}\right)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\\
&\quad +\rho_i\|\left(\frac{f_i(x_0)}{(g_i(x_0))^2}\right)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\Big]\\
&= \frac{g_i(x_0)}{g_i(x)}\Big[\omega_i^T\eta(x,x_0)+\rho_i\|\left(\frac{1}{g_i(x_0)}\right)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\\
&\quad +\rho_i\|\left(\frac{f_i(x_0)}{(g_i(x_0))^2}\right)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\Big]\\
&\geqq \frac{g_i(x_0)}{g_i(x)}\Big[\omega_i^T\eta(x,x_0)+\rho_i\|\left(\left(\frac{1}{g_i(x_0)}\right)^{\frac{1}{2}}+\left(\frac{(f_i(x_0))^{\frac{1}{2}}}{g_i(x_0)}\right)\right)\theta_i(x,x_0)\|^2\Big]\\
&= \frac{g_i(x_0)}{g_i(x)}\Big[\omega_i^T\eta(x,x_0)+\rho_i\|\left(\frac{(g_i(x_0))^{\frac{1}{2}}+(f_i(x_0))^{\frac{1}{2}}}{g_i(x_0)}\right)\theta_i(x,x_0)\|^2\Big]\\
&= \frac{g_i(x_0)}{g_i(x)}\Big[\omega_i^T\eta(x,x_0)+\rho_i\|\left(\frac{1+\left(\frac{f_i(x_0)}{g_i(x_0)}\right)^{\frac{1}{2}}}{(g_i(x_0))^{\frac{1}{2}}}\right)\theta_i(x,x_0)\|^2\Big].
\end{aligned}$$

Since $1+\left(\frac{f_i(x_0)}{g_i(x_0)}\right)^{\frac{1}{2}} \geqq 1$, $i=1,\ldots,p$, we have for any $\omega_i\in\partial\left(\frac{f_i}{g_i}\right)(x_0)$,

$$\alpha_i(x,x_0)\left(\frac{f_i(x)}{g_i(x)}-\frac{f_i(x_0)}{g_i(x_0)}\right)\geqq\frac{g_i(x_0)}{g_i(x)}\Big[\omega_i^T\eta(x,x_0)+\rho_i\|\left(\frac{1}{(g_i(x_0))^{\frac{1}{2}}}\right)\theta_i(x,x_0)\|^2\Big].$$

Thus, the function $\frac{f}{g}$ is (V, ρ)-invex at x_0, where

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)}\alpha_i(x, x_0), \quad \bar{\theta}_i(x, x_0) = \frac{1}{(g_i(x_0))^{\frac{1}{2}}}\theta_i(x, x_0).$$

□

2. Optimality theorems

Now, we establish the Kuhn-Tucker necessary and sufficient conditions for a solution of (GFP).

THEOREM 2.1. **(Kuhn-Tucker Necessary Optimality Theorem)** *Assume that f and $-g$ are regular. If x_0 is a solution of (GFP), and assume that $0 \notin \mathrm{co}\{\partial h_j(x_0) \mid j \in J(x_0)\}$, then there exist $\lambda_i \geq 0$, $i \in I(x_0) := \{i \mid \max\left\{\frac{f_i(x_0)}{g_i(x_0)} \mid i = 1, \ldots, p\right\} = \frac{f_i(x_0)}{g_i(x_0)}\}$, $\sum_{i \in I(x_0)} \lambda_i = 1$ and $\mu_j \geq 0$, $j = 1, \ldots, m$ such that*

$$0 \in \sum_{i \in I(x_0)} \lambda_i \partial\left(\frac{f_i}{g_i}\right)(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)$$

$$\text{and} \quad \sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

Proof. Let $\phi_i(x) = \frac{f_i(x)}{g_i(x)}$, $i = 1, \ldots, p$. Let x_0 be a solution of (GFP) and let $I(x_0) = \{i \mid \max\{\phi_i(x_0) \mid i = 1, \ldots, p\} = \phi_i(x_0)\}$. Then by Proposition 2.3.12 in [5] and Corollary 5.1.8 in [11], there exist $\mu_j \geqq 0$, $j = 1, \ldots, m$,

$$0 \in \mathrm{co}\{\partial\phi_i(x_0) \mid i \in I(x_0)\} + \sum_{j=1}^{m} \mu_j \partial h_j(x_0) \tag{2.1}$$

$$\text{and} \quad \mu_j h_j(x_0) = 0,$$

where co$A$ is the convexhull of the set A. By Lemma 1.2,

$$\begin{aligned} \partial\phi_i(x_0) =& \frac{g_i(x_0)\partial f_i(x_0) - \partial g_i(x_0) f_i(x_0)}{(g_i(x_0))^2} \\ =& \partial\left(\frac{f_i}{g_i}\right)(x_0), \end{aligned}$$

and hence from (2.1), there exist $\lambda_i \geq 0$, $i \in I(x_0)$, $\sum_{i\in I(x_0)} \lambda_i = 1$ and $\mu_j \geq 0$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i\in I(x_0)} \lambda_i \partial \left(\frac{f_i}{g_i}\right)(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0)$$
$$\text{and} \quad \sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

□

Corollary 2.2. *Let $f := (f_1, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1, \cdots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \cdots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. If x_0 is a solution of (GFP), and assume that $0 \notin \mathrm{co}\{\nabla h_j(x_0) \mid j \in J(x_0)\}$, then there exist $\lambda_i \geq 0$, $i \in I(x_0) := \{i \mid \max\left\{\frac{f_i(x_0)}{g_i(x_0)} \mid i = 1, \cdots, p\right\} = \frac{f_i(x_0)}{g_i(x_0)}\}$, $\sum_{i\in I(x_0)} \lambda_i = 1$ and $\mu_j \geq 0$, $j = 1, \ldots, m$ such that*

$$\sum_{i\in I(x_0)} \lambda_i \nabla \left(\frac{f_i(x_0)}{g_i(x_0)}\right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,$$
$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

Theorem 2.3. *(**Kuhn-Tucker Sufficient Optimality Theorem**) Assume that f and $-g$ are regular. Let x_0 be a feasible solution of (GFP). Suppose that there exist $\lambda_i \geqq 0$, $i \in I(x_0)$, $\sum_{i\in I(x_0)} \lambda_i = 1$ and $\mu_j \geqq 0$, $j = 1, \ldots, m$ such that*

$$0 \in \sum_{i\in I(x_0)} \lambda_i \partial \left(\frac{f_i}{g_i}\right)(x_0) + \sum_{j=1}^{m} \mu_j \partial h_j(x_0) \tag{2.2}$$
$$\text{and} \quad \sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

If $f(\cdot)$ and $-g(\cdot)$ are (V, ρ)-invex at x_0, and h is η-invex at x_0 with respect to the same η, and $\sum_{i\in I(x_0)} \lambda_i \rho_i \|\bar{\theta}_i(x, x_0)\|^2 \geqq 0$, then x_0 is a solution of (GFP).

Proof. Suppose that x_0 is not a solution of (GFP). Then there exist a feasible solution x of (GFP) such that

$$\max_{1\leqq i\leqq p} \frac{f_i(x_0)}{g_i(x_0)} > \max_{1\leqq i\leqq p} \frac{f_i(x)}{g_i(x)}.$$

Then

$$\frac{f_i(x_0)}{g_i(x_0)} > \frac{f_i(x)}{g_i(x)}, \text{for all } i \in I(x_0),$$

and hence $\bar{\alpha}_i(x, x_0) > 0$,

$$\bar{\alpha}_i(x, x_0) \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \right] < 0.$$

Since $f(\cdot)$ and $-g(\cdot)$ are (V, ρ)-invex and regular at x_0, by Theorem 1.3, we have for any $w_i \in \partial \left(\frac{f_i}{g_i} \right)(x_0)$, $i \in I(x_0)$

$$w_i \eta(x, x_0) + \rho_i \|\bar{\theta}(x, x_0)\|^2 < 0.$$

Hence, there exist $\lambda_i \geqq 0$, $i \in I(x_0)$, $\sum_{i \in I(x_0)} \lambda_i = 1$ such that

$$\sum_{i \in I(x_0)} \lambda_i w_i \eta(x, x_0) + \sum_{i \in I(x_0)} \lambda_i \rho_i \|\bar{\theta}(x, x_0)\|^2 < 0.$$

Since $\sum_{i \in I(x_0)} \lambda_i \rho_i \|\bar{\theta}(x, x_0)\|^2 \geqq 0$,

$$\sum_{i \in I(x_0)} \lambda_i w_i \eta(x, x_0) < 0,$$

and so, it follows from (2.2) that there exist $\nu_j \in \partial h_j(x_0)$, $j = 1, \ldots, m$ such that

$$\sum_{j=1}^{m} \mu_j \nu_j \eta(x, x_0) > 0.$$

Then, by the η-invexity of h, we have

$$\sum_{j=1}^{m} \mu_j h_j(x) > \sum_{j=1}^{m} \mu_j h_j(x_0).$$

Since $\sum_{j=1}^{m} \mu_j h_j(x_0) = 0$, we have $\sum_{j=1}^{m} \mu_j h_j(x) > 0$, which is a contradiction since $\mu_j \geqq 0$, $j = 1, \ldots, m$ and x is a feasible solution of (GFP). Consequently, x_0 is a solution of (GFP). □

3. Duality theorems

Now, we propose the following Mond-Weir type dual problem (DGFP):

$$\text{(DGFP)} \quad \text{Maximize} \quad \max\left\{\frac{f_i(u)}{g_i(u)} \mid i=1,\dots,p\right\}$$

$$\text{(3.1)} \quad \text{subject to} \quad 0 \in \sum_{i\in I(u)} \lambda_i \partial\left(\frac{f_i}{g_i}\right)(u) + \sum_{j=1}^{m} \mu_j \partial h_j(u)$$

$$\sum_{j=1}^{m} \mu_j h_j(u) = 0,$$

$$\lambda_i \geqq 0,\ i\in I(u),\ \sum_{i\in I(u)} \lambda_i = 1,\ \mu_j \geqq 0,\ j=1,\dots,m.$$

Now we show that the following weak duality theorem holds between (GFP) and (DGFP).

THEOREM 3.1. (**Weak Duality**) *Assume that f and $-g$ are regular. Let x be a feasible for (GFP) and let (u,λ,μ) be feasible for (DGFP). Assume that $f(\cdot)$ and $-g(\cdot)$ are (V,ρ)-invex at u, and let h is η-invex at u with respect to the same η, and $\sum_{i\in I(u)} \lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2 > 0$. Then the following holds:*

$$\max\left\{\frac{f_i(x)}{g_i(x)} \mid i=1,\dots,p\right\} \geqq \max\left\{\frac{f_i(u)}{g_i(u)} \mid i=1,\dots,p\right\}.$$

Proof. Let x be any feasible for (GFP) and let (u,λ,μ) be any feasible for (DGFP). Then we have

$$\sum_{j=1}^{m} \mu_j h_j(x) \leqq 0 \leqq \sum_{j=1}^{m} \mu_j h_j(u).$$

By the η-invexity of $h_j(u)$, $j=1,\dots,m$, there exists $\nu_j^* \in \partial h_j(u),\ j = 1,\cdots,m$ such that

$$\sum_{j=1}^{m} \mu_j \nu_j^* \eta(x,u) \leqq 0.$$

Using (3.1), we have there exists $w_i^* \in \partial\left(\frac{f_i}{g_i}\right)(u),\ i\in I(u)$,

$$\text{(3.2)} \qquad \sum_{i\in I(u)} \lambda_i\ w_i^* \eta(x,u) \geqq 0.$$

Now suppose that

$$\max\left\{\frac{f_i(x)}{g_i(x)} \mid i=1,\dots,p\right\} < \max\left\{\frac{f_i(u)}{g_i(u)} \mid i=1,\dots,p\right\}.$$

Then

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}, \text{ for all } i \in I(u).$$

By Theorem 1.3, we have there exists $w_i^* \in \partial\left(\frac{f_i}{g_i}\right)(u)$, $i \in I(u)$ such that

$$\begin{aligned} 0 > \bar{\alpha}_i(x,u)&\left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)}\right] \\ &\geqq w_i^*\eta(x,u) + \rho_i\|\bar{\theta}_i(x,u)\|^2. \end{aligned}$$

By using $\lambda_i \geqq 0$, $i \in I(u)$, we have,

$$\sum_{i\in I(u)} \lambda_i w_i^*\eta(x,u) + \sum_{i\in I(u)} \lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2 < 0.$$

Since $\sum_{i\in I(u)} \lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2 \geqq 0$, we have

$$\sum_{i\in I(u)} \lambda_i w_i^*\eta(x,u) < 0,$$

which contradicts (3.2). Hence the result holds. □

Now we give a strong duality theorem which holds between (GFP) and (DGFP).

THEOREM 3.2. (**Strong Duality**) *If $\bar{x}$ is a solution of (GFP) and suppose that $0 \notin \mathrm{co}\{\partial h_j(x_0) \mid j \in J(x_0)\}$. Then there exist $\bar{\lambda} \in \mathbb{R}^p$ and $\bar{\mu} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (DGFP). Moreover if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP).*

Proof. By Theorem 2.1, there exist $\bar{\lambda}_i \geq 0$, $i \in I(\bar{x}) := \{i \mid \max\{\frac{f_(\bar{x})}{g_i(\bar{x})} \mid i = 1,\dots,p\} = \frac{f_(\bar{x})}{g_i(\bar{x})}\}$, $\sum\limits_{i\in I(\bar{x})} \bar{\lambda}_i = 1$ and $\bar{\mu}_j \geq 0$, $j = 1,\dots,m$ such that

$$\begin{aligned} &0 \in \sum_{i\in I(\bar{x})} \bar{\lambda}_i \partial\left(\frac{f_i}{g_i}\right)(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial h_j(\bar{x}) \\ &\text{and} \quad \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0. \end{aligned}$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible for (DGFP). On the other hand, by weak duality (Theorem 3.1),

$$\max \left\{ \frac{f_i(\bar{x})}{g_i(\bar{x})} \mid i = 1, \cdots, p \right\} \geq \max \left\{ \frac{f_i(u)}{g_i(u)} \mid i = 1, \cdots, p \right\}$$

for any (DGFP) feasible solution (u, λ, μ). Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP). □

4. Conclusions

This paper is concerned with optimality conditions and duality theorems for fractional optimization problems involving locally Lipschitz functions. Using Clarke's generalized subdifferential, we gave necessary and sufficient optimality theorems for the problems. The sufficient optimality conditions were verified under generalized invexity conditions on involved functions. The Mond-Weir dual problems were formulated, and then duality theorems were established, that is, weak and strong duality theorems for the nondifferentiable fractional optimization problems.

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